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AN ASYMPTOTIC INVESTIGATION OF THE SMALL STRAIN THEORY OF SHELLS

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Abstract—A small strain theory of shells is derived from the three dimensional equations of equilibrium and compatibility by the usual asymptotic approach. The highest order equations are seen to agree with those of Koiter [9] and John [12].

In the derivation of these equations three small parameters appear and different special theories are obtained when different assumptions on the relative magnitudes of the parameters are made.

In cases where either the extensional stresses or the bending stresses are dominant, special refined theories are required. These are obtained in the linear case and are found to agree with John's refined interior equations [13, 14].

1. INTRODUCTION

THE asymptotic approach to the linear theory of shells is now well established (e.g. Friedrichs and Dressler [1], Green [2, 3], Goldenweizer [4]) and some specific problems have been treated (e.g. Riess [5], Westbrook [6, 7]).

The asymptotic methods have also been applied to the non-linear theory of plates by Ebcioglu and Habip [8] but does not seem to have been applied to shells.

In this paper we carry out this work to obtain shell equations which are valid in the interior of the shell, that is away from the edges.

The boundary layer or edge effects we postpone for a later study.

The work here is carried out under the assumption of small strains. Following Koiter [9] and Chien [10] we do not introduce the displacements but make use of the compatibility equations instead. This has the advantage that no additional assumptions on the magnitudes of the displacement gradients are required but suffers the disadvantages that it does not allow an easy formulation of displacement boundary conditions and that it necessitates the use of the Cauchy stress tensor which does not in general lend itself easily to the formulation of boundary conditions of traction (see Truesdell and Noll [11, p. 125]). For the present purposes, however, it is assumed that we are dealing with traction boundary conditions which may be expressed in terms of the Cauchy stress tensor.

Comparisons of the equations are made with Koiter [9], who assumes a state of approximately plane stress, and with the rigorous results of John [12]. Agreement is obtained when we make certain assumptions on the loading. These assumptions are very natural and would be satisfied under almost any loadings of engineering interest.

The appeal of the asymptotic approach is that the assumptions seem natural and that the development is then consistent.

It will be assumed throughout that the material is isotropic and homogeneous. The usual linear stress strain relations are used with an error which is of the order of the square of the strains. Since these second order terms are not considered explicitly the resulting theory contains only the leading terms in the asymptotic expansions. It should, however, be clear from the present work, how to obtain higher order approximations if they are required.

We consider a thin shell of undeformed normal thickness $2h$. Here we consider h as constant but this is not essential to the method and a variation in h can be taken into account provided that partial derivatives of h do not become large compared to h itself. The shell coordinates and the curvature tensor of the undeformed middle surface are scaled together with the strains and it is found that the differential equations of the elastic theory contain three parameters all of which are assumed to be small; ε, a measure of the magnitudes of the strain, $\theta = h/R$, where R is related to the curvature tensor and $\delta = h/L$, where L is a typical length related to the dimensions of the shell and the wavelength of the boundary data.

The final equations given here contain various ratios of these small parameters with the neglected terms always smaller than some term remaining in the equations. Different special sets of shell equations are obtained when different assumptions are made on the relative magnitudes of the small parameters.

Since this work was completed, two papers by Koiter [15, 16] have appeared which show that in the linear case, the root mean square error in the energy between the classical shell solution and the solution of the three dimensional problem is of the order $\theta+\delta^2$. It seems probable that a similar error estimate would be found in the present case for the linear problem if equations (4.6) and (4.7) are used.

2. DESCRIPTION OF THE SHELL, THE STRAINS AND THE EQUATIONS OF COMPATIBILITY

In the undeformed state the shell has two curved surfaces or faces Σ_h and Σ_{-h} and a lateral surface or edge B. The faces are at a constant distance h from a middle surface Σ_0, one face on each side of Σ_0. The particles are assigned Lagrangian coordinates u^1, u^2, u^3 where u^1, u^2 are coordinates on the undeformed middle surface Σ_0 and points on a normal to Σ_0 have the same u^1, u^2 coordinates. u^3 is the distance from Σ_0 measured along a normal to Σ_0. The faces Σ_h and Σ_{-h} of the shell are then given by $u^3 = h, u^3 = -h$ respectively. Since we are going to give special attention to the coordinate u^3 we will use Greek letters for indices having only the values 1, 2 and Latin letters for those having values 1, 2, 3. The notation $f_{,i}$ for the partial derivative of f with respect to u^i will be used. We suppose that in the undeformed state the particles have cartesian coordinates $X^i = X^i(u^1, u^2, u^3)$ and that in the deformed state they have cartesian coordinates $x^i = x^i(u^1, u^2, u^3)$. The fundamental tensors in the undeformed and deformed states are represented by G_{ij} and g_{ij} respectively, so that

$$G_{ij} = X^k_{,i}X^k_{,j}, \qquad g_{ij} = x^k_{,i}x^k_{,j}.$$

We give preference to the undeformed metric and all raising and lowering of suffices and covariant differentiation will refer to this metric.

The strains ε_{ij} are defined by

$$\varepsilon_{ij} = \tfrac{1}{2}(g_{ij}-G_{ij}). \tag{2.1}$$

With the given choice of coordinates we note that

$$\left.\begin{aligned} G_{\alpha\beta} &= A_{\alpha\beta}-2u^3B_{\alpha\beta}+(u^3)^2B^\gamma_\alpha B_{\gamma\beta} \\ G_{\alpha 3} &= 0, \qquad G_{33} = 1, \end{aligned}\right\} \tag{2.2}$$

where $A_{\alpha\beta}, B_{\alpha\beta}$ are the coefficients of the first and second fundamental forms of the undeformed mid-surface Σ_0.

In order to form covariant derivatives with respect to the deformed metric, we note that the Christoffel symbols $\overline{\{^{i}_{jk}\}}$ of the deformed metric are such that

$$\overline{\{^{i}_{jk}\}} = \{^{i}_{jk}\}+C^i_{jk} \tag{2.3}$$

where $\{^{i}_{jk}\}$ are the corresponding Christoffel symbols in the undeformed metric and

$$C^i_{jk} = \bar{g}^{ir}(\varepsilon_{rj;k}+\varepsilon_{kr;j}-\varepsilon_{kj;r})$$

where $\bar{g}^{ir}$ is such that $\bar{g}^{ir}g_{rj} = \delta^i_j$ and ";" denotes covariant differentiation with respect to the undeformed metric.

The equations of compatibility which express the fact that the Riemman Christoffel curvature tensor for both the undeformed and the deformed metrics must be zero are found to be

$$\varepsilon_{hk;ij}+\varepsilon_{ij;hk}-\varepsilon_{hj;ik}-\varepsilon_{ik;hj}+(G_{ab}+2\varepsilon_{ab})(C^b_{hk}C^a_{ij}-C^b_{hj}C^a_{ik}) = 0. \tag{2.4}$$

Of these equations only six are independent.

3. THE STRESSES AND THE EQUATIONS OF EQUILIBRIUM

We use the Cauchy stress tensor whose components, referred to the u^i system, are denoted by t^{ij}. This means that if $\mathrm{d}A$ is an element of area of the deformed body with direction cosines n_l referred to the deformed cartesian x^l system then the resultant force on $\mathrm{d}A$ referred to the x^l system is

$$t^{ij}x^k_{,i}x^l_{,j}n_l\,\mathrm{d}A.$$

The equation of equilibrium may be written

$$t^{ij}_{;j}+C^i_{jk}t^{jk}+C^k_{jk}t^{ij} = 0. \tag{3.1}$$

The body force has been omitted for convenience. No difficulties arise from its inclusion.

For an isotropic homogeneous material we will have the constitutive equation:

$$t^{ij} = 2\mu\left[\varepsilon^{ij}+\frac{\nu}{1-\nu}G^{ij}\varepsilon^k_k\right]+0(\varepsilon^2), \tag{3.2}$$

where $0(\varepsilon^2)$ denotes terms whose magnitude is of the order of the square of the strains and where μ is the elastic shear modulus and ν a modified Poisson's ratio given in terms of the Poisson's ratio σ by the equation $(1+\nu)(1-\sigma) = 1$.

It should be noted here that t^{ij} is a tensor in the deformed coordinates and hence raising and lowering of the suffices is performed by means of the g_{ij}.

The strain tensor ε_{ij} however is taken to be a tensor in the undeformed coordinates and raising and lowering of suffices is performed by means of G_{ij}. The error involved in raising the suffices in stress strain relation (3.2) is of the order of ε^2 and hence may be absorbed in the higher order terms of the stress strain relations.

We assume boundary conditions $t^{i3} = (\mu/2)[Q^i \pm P^i]$ on the surfaces $u_3 = \pm h$ of the deformed body.

4. THE SHELL EQUATIONS

We now scale the coordinates and the second fundamental tensor of Σ_0 in the following manner. Let $\xi^\alpha = u^\alpha/L$ where L is a typical length whose maximum value is determined by the boundary data and the shell geometry and let $\zeta = u^3/h$. In scaling the second fundamental tensor we follow John [12]. Let P_0 be a point of Σ_0 and let the Cartesian coordinate system X^i be chosen so that P_0 is the origin and the X^3 axis is normal to Σ_0 at P_0. In the neighbourhood of P_0, Σ_0 is given by an equation $X^3 = f(X_1, X_2)$ where f and its first partial derivatives vanish at the origin. We define the length R such that

$$|f_{,\alpha\beta}| < R^{-1}, \quad |f_{,\alpha\beta\gamma}| < R^{-2}, \text{ etc.}$$

for any choice P_0 of Σ_0. It may then be shown, John [12], that $B_{\alpha\beta}$ is of order R^{-1} and also that $A_{\alpha\beta,\gamma}$ is of order R^{-1}.

We now define two small parameters $\theta = h/R$, $\delta = h/L$ and a scaled curvature tensor

$$K_{\alpha\beta} = RB_{\alpha\beta}.$$

From this point $f_{,\alpha}$ will denote the partial derivative of f with respect to ξ^α and $f_{,\zeta}$ the partial derivative with respect to ζ.

We note here the leading terms of the Christoffel symbols of the undeformed metric.

$$\left\{ {\alpha \atop \beta\gamma} \right\} = \frac{1}{L}\left[\frac{\theta}{\delta}\Gamma^\alpha_{\beta\gamma} + O(\theta^2/\delta)\right]$$

where

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2}RA^{\alpha\delta}(A_{\beta\delta,\gamma} + A_{\gamma\delta,\beta} - A_{\beta\gamma,\delta}) \tag{4.1}$$

is a scaled Christoffel symbol formed in the undeformed surface metric $A_{\alpha\beta}$ of Σ_0.

$$\left\{ {3 \atop \alpha\beta} \right\} = \frac{1}{R}[K_{\alpha\beta} + O(\theta)]$$

$$\left\{ {\alpha \atop 3\beta} \right\} = -\frac{1}{R}[K^\alpha_\beta + O(\theta)]$$

$$\left\{ {3 \atop 3k} \right\} = \left\{ {\alpha \atop 33} \right\} = 0. \tag{4.1}$$

The raising and lowering of the indices α, β here is by use of the surface metric $A_{\alpha\beta}$.

The equations are scaled under the usual assumptions of the asymptotic method; that the order of magnitude of any terms is not increased when they are differentiated with respect to ξ_α or ζ, or

$$|f_{ij,\alpha\beta\ldots\sigma\zeta\ldots\zeta}/f_{ij}|$$

remains bounded as h tends to zero and where f_{ij} denotes any stress or strain component.

We now follow John [12] and define the small parameter $\gamma = \max[\delta, \theta^{\frac{1}{2}}, \varepsilon^{\frac{1}{2}}]$. There are however occasions when it becomes convenient to indicate some errors in terms of δ as well as γ. Writing $t^{ij} = \mu s^{ij}$ we obtain from the equations of equilibrium (3.1)

$$s^{\alpha 3}_{,\zeta} = O(\varepsilon\delta \text{ or } \varepsilon\gamma^2)$$

$$s^{33}_{,\zeta} = O(\varepsilon\gamma^2).$$

Making use of the boundary conditions (3.3) we see that

$$s^{\alpha 3} = \tfrac{1}{2}Q^{\alpha} + O(\varepsilon\delta \text{ or } \varepsilon\gamma^2), \qquad s^{33} = \tfrac{1}{2}Q^3 + O(\varepsilon\gamma^2)$$
$$P^{\alpha} = O(\varepsilon\delta \text{ or } \varepsilon\gamma^2), \qquad P^3 = O(\varepsilon\gamma^2).$$

The condition on P^i is one of the criteria which must be satisfied in order that a small strain theory is applicable. In almost all surface loadings of engineering interest, P^i and Q^i are of the same order of magnitude. From this point we will assume that this is so and hence we have

$$\begin{aligned} s^{\alpha 3} &= O(\varepsilon\delta \text{ or } \varepsilon\gamma^2), \ \tfrac{1}{2}Q^{\alpha} \text{ and } \tfrac{1}{2}P^{\alpha} \text{ are } O(\varepsilon\delta \text{ or } \varepsilon\gamma^2) \\ s^{33} &= O(\varepsilon\gamma^2) \qquad Q^3 \text{ and } P^3 \text{ are } O(\varepsilon\gamma^2). \end{aligned} \tag{4.2}$$

It is worthwhile perhaps to point out that this assumption that Q^i is of the same order of magnitude as P^i is not necessary and the theory may be worked out without it. The resulting equations will differ from those of authors who use other approaches indicating that those theories are probably incorrect if such an admittedly unlikely loading occurred.

The equations (4.2) are in a sense equivalent to Koiter's assumption of approximately plane stress although we shall see that some slightly different results will emerge later.

In the case of an isotropic and homogenous material we have

$$\begin{aligned} \varepsilon^{\alpha 3} &= O(\varepsilon\delta \text{ or } \varepsilon\gamma^2) \\ \varepsilon^{33} = \varepsilon^3_3 &= -\nu\varepsilon^{\gamma}_{\gamma} + O(\varepsilon\gamma^2). \end{aligned} \tag{4.3}$$

We make use of these results in three of the equations of compatibility (2.4) with $h = \alpha$, $i = 3, j = \beta, k = 3$. In the scaled coordinates we obtain

$$\varepsilon_{\alpha\beta,\zeta\zeta} = O(\varepsilon\gamma^2)$$

or

$$\varepsilon_{\alpha\beta} = E_{\alpha\beta} + \zeta W_{\alpha\beta} + O(\varepsilon\gamma^2). \tag{4.4}$$

The tensors $E_{\alpha\beta}$ and $W_{\alpha\beta}$ will be treated as surface tensors so that raising of suffices will be performed by the use of the surface metric $A^{\alpha\beta}$. Suffices on the $\varepsilon_{\alpha\beta}$ tensor however are raised by use of the metric $G^{\alpha\beta}$.

Making use of the results (4.3) and (4.4) we now calculate the tensors C^i_{jk}. We find that

$$hC^{\alpha}_{\beta\gamma} = O(\varepsilon\delta \text{ or } \varepsilon\gamma^2), \qquad hC^{\alpha}_{3\beta} = W^{\alpha}_{\beta} + O(\varepsilon\gamma^2)$$
$$hC^{\alpha}_{33} = O(\varepsilon\delta \text{ or } \varepsilon\gamma^2), \qquad hC^{3}_{\beta\gamma} = -W_{\beta\gamma} + O(\varepsilon\gamma^2)$$
$$hC^{3}_{\beta 3} = O(\varepsilon\delta \text{ or } \varepsilon\gamma^2), \qquad hC^{3}_{33} = -\nu W^{\gamma}_{\gamma} + O(\varepsilon\gamma^2).$$

We substitute these in (3.1) and make use of (4.1) to obtain the following equilibrium equations

$$s^{\alpha 3}_{,\zeta} + \delta s^{\alpha\beta}|_{\beta} = O(\varepsilon\gamma^3\delta \text{ or } \varepsilon\gamma^4) \tag{4.5a}$$

$$s^{33}_{,\zeta} + \delta s^{3\beta}|_{\beta} + \theta K_{\beta\gamma}s^{\beta\gamma} - W_{\beta\gamma}s^{\beta\gamma} = O(\varepsilon\gamma^4) \tag{4.5b}$$

Here "|" denotes scaled covariant differentiation with respect to the surface metric, $s^{\alpha\beta}$ being treated as a second order surface tensor and $s^{\alpha 3}$ as a surface vector, in fact

$$s^{\alpha\beta}|_{\beta} = s^{\alpha\beta}_{,\beta} + \frac{\theta}{\delta}\Gamma^{\alpha}_{\gamma\beta}s^{\gamma\beta} + \frac{\theta}{\delta}\Gamma^{\beta}_{\gamma\beta}s^{\alpha\gamma}$$

$$s^{3\beta}|_{\beta} = s^{3\beta}_{,\beta} + \frac{\theta}{\delta}\Gamma^{\beta}_{\gamma\beta}s^{3\gamma}.$$

Using the constitutive equation (3.2) and (4.4) we have

$$s^{\alpha\beta} = N^{\alpha\beta} + \zeta M^{\alpha\beta} + O(\varepsilon\gamma^2) \tag{4.6}$$

where

$$N^{\alpha\beta} = 2[E^{\alpha\beta} + \nu E^{\gamma}_{\gamma} A^{\alpha\beta}]$$

$$M^{\alpha\beta} = 2[W^{\alpha\beta} + \nu W^{\gamma}_{\gamma} A^{\alpha\beta}].$$

$N^{\alpha\beta}$, $M^{\alpha\beta}$ are thus surface tensors. Substituting (4.6) into (4.5a), integrating with respect to ζ and using the boundary condition (3.3) we find

$$s^{\alpha 3} = \tfrac{1}{2}Q^{\alpha} + \tfrac{1}{2}P^{\alpha}\zeta + \tfrac{1}{2}\delta(1-\zeta^2)M^{\alpha\beta}|_{\beta} + O(\varepsilon\gamma^2\delta \text{ or } \varepsilon\gamma^4) \tag{4.7a}$$

and

$$\delta N^{\alpha\beta}|_{\beta} + \tfrac{1}{2}P^{\alpha} = O(\varepsilon\gamma^2\delta \text{ or } \varepsilon\gamma^4). \tag{4.8a}$$

We now use this new expression for $s^{\alpha 3}$ in (4.5b) integrate with respect to ζ and use the boundary condition (3.3) to obtain,

$$s^{33} = \tfrac{1}{2}Q^3 + \tfrac{1}{2}\zeta P^3 + \frac{\delta}{4}(1-\zeta^2)P^{\gamma}|_{\gamma} + \tfrac{1}{2}(1-\zeta^2)K_{\beta\gamma}M^{\beta\gamma}$$
$$-\tfrac{1}{2}(1-\zeta^2)W_{\beta\gamma}M^{\beta\gamma} - \frac{\delta^2}{6}\zeta(1-\zeta^2)M^{\alpha\beta}|_{\alpha\beta} + O(\varepsilon\gamma^4) \tag{4.7b}$$

and

$$\tfrac{1}{3}\delta^2 M^{\alpha\beta}|_{\alpha\beta} + \theta K_{\beta\gamma}N^{\beta\gamma} - W_{\beta\gamma}N^{\beta\gamma} + \tfrac{1}{2}P^3 + \tfrac{1}{2}\delta Q^{\gamma}|_{\gamma} = O(\varepsilon\gamma^4) \tag{4.8b}$$

The other equations for our theory are obtained from the remaining three compatibility equations. These equations will now be polynomials in ζ. It is found however that when the constant terms are equated to zero for all three equations then the other terms become identities. This may be expected because of certain identities obtained by differentiation and combination of the compatibility equations. The constant terms are in fact just those which would be obtained from the Gauss Codazzi equations of the deformed surface. They are

$$\delta^2[E_{12|12} + E_{21|21} - E_{11|22} - E_{22|11}]$$
$$+\theta[K_{11}W_{22} + K_{22}W_{11} - 2K_{12}W_{12}] - [W_{11}W_{22} - W_{12}^2] = O(\varepsilon\gamma^4) \tag{4.9a}$$

$$\delta(W_{12|2} - W_{22|1}) = O(\varepsilon\gamma^2\delta) \tag{4.9b}$$

$$\delta(W_{21|1} - W_{11|2}) = O(\varepsilon\gamma^2\delta) \tag{4.9c}$$

They may be written as

$$e^{\alpha\beta} e^{\sigma\tau}[\delta^2 E_{\alpha\tau|\sigma\beta} - K_{\alpha\tau}W_{\sigma\beta} + \tfrac{1}{2}W_{\alpha\tau}W_{\sigma\beta}] = O(\varepsilon\gamma^4) \tag{4.10a}$$

and

$$\delta W^{\alpha}_{\beta|\gamma} - \delta W^{\alpha}_{\gamma|\beta} = O(\varepsilon\gamma^2\delta) \tag{4.10b}$$

where $e^{\alpha\beta}$ is the surface permutation tensor.

If we note that the $W_{\alpha\beta}$ used here is not the change in curvature used by Koiter and John, but that in fact the deformed curvature tensor $b_{\alpha\beta}$ is given by

$$\begin{aligned} b_{\alpha\beta} &= [(\bar{g}^{33})^{-\frac{1}{2}}\left\{ {3 \atop \alpha\beta} \right\}]_{u_3=0} \\ &= B_{\alpha\beta} - \frac{W_{\alpha\beta}}{h} + \frac{1}{h} 0(\varepsilon\gamma^2). \end{aligned}$$

It is then seen that our equations (4.8a), (4.8b), (4.9a), (4.9b), (4.9c) are in agreement with John's lowest order interior equations when the surface loads are zero.

There is a slight difference between our equations and Koiter's simplified equation when surface loading is taken into account. It is the appearance of the term $\delta Q^{\alpha}|_{\alpha}$ in our equation (4.8b). The Q^{α} term cannot appear in Koiter's theory because he only considers the resultant shear on the middle surface. The term is always of order $\varepsilon\delta^2$ or less and will be negligible in many engineering loadings.

As Koiter [9] points out the shell equation (4.8), (4.9) are valid under a wide range of loadings. Specifically, they are valid provided that both

$$W \gg \varepsilon\gamma^2 \text{ and } E \gg \varepsilon\gamma^2$$

where one of E or W must be of order ε.

In the remaining two cases $W \leq \varepsilon\gamma^2$ or $E \leq \varepsilon\gamma^2$ we will need to make refinements to our equations.

5. REFINED THEORIES

In obtaining refined equations we consider only theories which are essentially linear, that is we take $\varepsilon \leq \gamma^4$. This is done because otherwise we require to know the $0(\varepsilon^2)$ terms in the stress–strain relations and to take into account difficulties in raising and lowering suffices in the deformed as well as the undeformed metric. Such refined equations could be obtained but it is felt that the effort involved is too great for the results particularly in view of the fact that non-linear constitutive equations have not been developed to any great extent.

Case 1

We assume a linear theory $\varepsilon \leq \gamma^4$ and $W \leq \varepsilon\gamma^2$.

The equations (4.8a), (4.8b) do not need refinement and remain as

$$\delta N^{\alpha\beta}|_{\beta} + \tfrac{1}{2}P^{\alpha} = 0(\varepsilon\gamma^2\delta \text{ or } \varepsilon\gamma^4) \tag{5.1a}$$

$$\theta K_{\alpha\beta}N^{\alpha\beta} + \tfrac{1}{2}P^3 + \tfrac{1}{2}\delta Q^{\alpha}|_{\alpha} = 0(\varepsilon\gamma^4). \tag{5.1b}$$

To obtain further refinement of the compatibility equations we first return to the equations for which $h = \alpha$, $k = 3$, $i = 3$, $j = \beta$ and use the new approximations (4.7a), (4.7b) for $s^{\alpha 3}$ and s^{33}. We find that after integration

$$\begin{aligned} \varepsilon_{\alpha\beta} = {} & E_{\alpha\beta} + \delta W_{\alpha\beta} + \tfrac{1}{2}\delta\zeta[Q_{\alpha|\beta} + Q_{\beta|\alpha}] + \frac{\delta}{4}\zeta^2[P_{\alpha|\beta} + P_{\beta|\alpha}] - \tfrac{1}{2}\theta\zeta^2[K^{\sigma}_{\alpha}W_{\sigma\beta} + K^{\sigma}_{\beta}W_{\sigma\alpha}] \\ & + \tfrac{1}{2}\nu\theta\zeta^2 K_{\alpha\beta}W^{\sigma}_{\sigma} + \tfrac{1}{2}\nu\zeta^2\delta^2 E^{\sigma}_{\sigma|\alpha\beta} + \tfrac{1}{3}\nu\zeta^3\delta^2 W^{\sigma}_{\sigma|\alpha\beta} \\ & - \tfrac{1}{3}\delta^2\zeta^3[M^{\sigma}_{\alpha|\sigma\beta} + M^{\sigma}_{\beta|\sigma\alpha}] + 0(\varepsilon\gamma^4). \end{aligned} \tag{5.2}$$

We use (5.1) together with (4.7a), (4.7b) to obtain the refined compatibility equations

$$e^{\alpha\beta}e^{\sigma\tau}[\delta^2 E_{\alpha\tau|\sigma\beta}-\theta K_{\alpha\tau}W_{\sigma\beta}-\tfrac{1}{2}\theta^2(1-2\nu)K_{\alpha\tau}K_{\sigma\beta}E^\gamma_\gamma] = O(\varepsilon\gamma^6). \tag{5.3a}$$

If we note that when $W = O(\varepsilon\gamma^2)$ then

$$h(b_{\alpha\beta}-B_{\alpha\beta}) = -W_{\alpha\beta}+\nu\theta K_{\alpha\beta}E^\sigma_\sigma+O(\varepsilon\gamma^4). \tag{5.4}$$

we see that this agrees with Koiter's exact equation. The other two compatibility equations become

$$\delta(W_{12|2}-W_{22|1})+\theta\delta[K^\tau_1 E_{\tau 2|2}-K^\tau_2 E_{1\tau|2}]+\theta\delta K^\tau_\tau[E_{12|2}-E_{22|1}] \\ +\nu\theta\delta[K_{22}E^\tau_{\tau|1}-K_{12}E^\tau_{\tau|2}] = O(\varepsilon\gamma^4\delta) \tag{5.3b}$$

$$\delta(W_{21|1}-W_{11|2})+\theta\delta[K^\tau_2 E_{\tau 1|1}-K^\tau_1 E_{\tau 2|1}]+\theta\delta K^\tau_\tau[E_{21|1}-E_{11|2}] \\ +\nu\theta\delta[K_{11}E^\tau_{\tau|2}-K_{12}E^\tau_{\tau|1}] = O(\varepsilon\gamma^4\delta) \tag{5.3c}$$

which are also in agreement with Koiter's exact equations when (5.4) is taken into account.

The equations (5.1) and (5.2) are the membrane shell equations. Equations (5.3a, b, c) are then equations for W.

As was noted by Koiter [9], if $\theta \leq \delta^2$ then the first term in (5.3a) becomes dominant and the system appears to be overdetermined suggesting that such an occurrence is unlikely. However if θ is of $O(\gamma^4)$ the equation (5.1b) is no longer valid and a consistent theory is possible. In fact we find that the equations become

$$\delta^2\, e^{\alpha\beta}\, e^{\sigma\tau}E_{\alpha\tau|\sigma\beta} = O(\varepsilon\gamma^6) \qquad \delta N^{\alpha\beta}|_\beta+\tfrac{1}{2}P^\alpha = O(\varepsilon\gamma^2\delta \text{ or } \varepsilon\gamma^4)$$

a generalized plane stress problem.

This is in agreement with an earlier result of Koiter [17] in the linear theory of shells.

Case 2

$$\varepsilon \leq \gamma^4, E \leq \varepsilon\gamma^2.$$

In this case the compatibility equations require little refinement and we obtain

$$\theta[K^\sigma_\tau W^\tau_\sigma - K^\sigma_\sigma W^\tau_\tau] = O(\varepsilon\gamma^4) \tag{5.4a}$$

$$\delta W^\alpha_{\beta|\gamma}-\delta W^\alpha_{\gamma|\beta} = O(\varepsilon\gamma^2\delta). \tag{5.4b}$$

To refine the equilibrium equations, we first make use of the improved approximation (5.2) for $\varepsilon_{\alpha\beta}$ and we obtain

$$\begin{aligned} s^\alpha_\beta = {} & N^\alpha_\beta+\zeta M^\alpha_\beta+\theta\zeta^2[3K^\alpha_\tau W^\tau_\beta-K^\tau_\beta W^\alpha_\tau]+\nu\theta\zeta^2K^\alpha_\beta W^\sigma_\sigma \\ & +\tfrac{1}{2}\delta\zeta^2[P^\alpha|_\beta+P_\beta|^\alpha]+\delta^\alpha_\beta[\tfrac{1}{2}\nu Q_3+\tfrac{1}{4}\nu\delta(1+3\zeta^2)P^\gamma|_\gamma \\ & +\nu\theta(1+\zeta^2)K^\tau_\sigma W^\sigma_\tau+\nu^2\theta K^\tau_\tau W^\sigma_\sigma] \\ & +\text{terms of } O(\varepsilon\delta^2) \text{ and odd in } \zeta+O(\varepsilon\gamma^4). \end{aligned} \tag{5.5}$$

The equilibrium equation (2.1) gives

$$\delta s^\beta_{\alpha|\beta}+s^3_{\alpha,\zeta}-\theta K^\sigma_\alpha[\tfrac{1}{2}Q_\sigma+\tfrac{1}{2}\zeta P_\sigma+\delta(1-\zeta^2)(1+\nu)W^\gamma_{\gamma|\sigma}]-\theta\zeta^2\delta K^\sigma_{\sigma|\tau}M^\tau_\alpha+\theta\zeta^2\delta K^\tau_{\beta|\alpha}M^\beta_\tau = O(\varepsilon\gamma^4\delta).$$

This is integrated with respect to ζ, and use is made of the boundary condition and the equations (5.4a), (5.4b) to obtain the following refined equation

$$\begin{aligned}&\delta N^{\alpha}_{\beta|\alpha}+\tfrac{1}{3}\theta\delta[K^{\alpha}_{\tau}W^{\tau}_{\beta}+K^{\tau}_{\beta}W^{\alpha}_{\tau}+\nu K^{\alpha}_{\beta}W^{\tau}_{\tau}+\delta^{\alpha}_{\beta}\nu(4+3\nu)K^{\sigma}_{\sigma}W^{\tau}_{\tau}]_{|\alpha}\\&\quad-\tfrac{2}{3}(1+\nu)\theta\delta K^{\sigma}_{\sigma}W^{\tau}_{\tau|\beta}-\tfrac{2}{3}\theta\delta K^{\sigma}_{\beta}W^{\tau}_{\tau|\sigma}+\tfrac{2}{3}\theta\delta K^{\sigma}_{\tau}W^{\tau}_{\sigma|\beta}\\&\quad+\tfrac{1}{2}P^{\alpha}+\tfrac{1}{2}\nu\delta Q^{3}|_{\beta}-\tfrac{1}{2}\theta K^{\sigma}_{\sigma}Q_{\beta}+\tfrac{1}{6}\delta^{2}(1+3\nu)P^{\sigma}|_{\sigma\beta}\\&\quad+\tfrac{1}{6}\delta^{2}P_{\beta}|^{\tau}_{\tau}=0(\varepsilon\gamma^{4}\delta).\end{aligned}\tag{5.6a}$$

This equation is seen to be in agreement with John's [13, 14] corrected refined equation when note is taken of the compatibility equation (5.4a). In the second equation of equilibrium, we find that the equilibrium equation becomes

$$\begin{aligned}&\delta s^{\alpha 3}|_{\alpha}-\theta K^{\sigma}_{\sigma}\left|\frac{1}{2}Q_{3}+\frac{1}{2}\zeta P_{3}+\frac{\theta}{2}(1-\zeta^{2})K_{\alpha\beta}M^{\alpha\beta}\right|+\theta K_{\alpha\beta}s^{\alpha\beta}\\&\quad-\theta^{2}\zeta^{2}K^{\tau}_{\alpha}K^{\beta}_{\tau}M^{\alpha}_{\beta}+s^{33}_{,\zeta}=0(\varepsilon\gamma^{6}).\end{aligned}$$

When this is integrated and use is made of the boundary conditions and the compatibility equations (4.4a), (4.4b) we obtain the following refined equation

$$\begin{aligned}&\tfrac{1}{3}\delta^{2}M^{\sigma\tau}|_{\sigma 1}+\theta K^{\tau}_{\sigma}N^{\sigma}_{\tau}+\tfrac{1}{2}P^{3}-\tfrac{1}{2}(1-\nu)\theta K^{\sigma}_{\sigma}Q^{3}+\tfrac{1}{2}\delta Q^{\sigma}|_{\sigma}\\&\quad+\tfrac{1}{3}\theta^{2}|3\nu K^{\tau}_{\alpha}K^{\alpha}_{\tau}W^{\sigma}_{\sigma}+\tfrac{4}{3}K^{\alpha}_{\beta}K^{\beta}_{\tau}W^{\tau}_{\alpha}-\nu(2-3\nu)K^{\gamma}_{\gamma}K^{\tau}_{\tau}W^{\sigma}_{\sigma}-(2-4\nu)K^{\sigma}_{\sigma}K^{\alpha}|=0(\varepsilon\gamma^{6})\end{aligned}\tag{5.6b}$$

This equation is also in complete agreement with John's [13] refined equation.

It is not immediately clear that Koiter's equations [9] agree with John's refined equations [13, 14] since great care must be taken in identifying the various terms appearing in the respective sets of equations. It was in Koiter's paper [18] that the comparison of these equations in certain special cases indicated clerical errors in John's original derivation [13] which were subsequently corrected [14]. More recent work by Koiter has established the equivalence of his equations and John's in the general case.

6. FURTHER SIMPLIFICATIONS

It is possible to make further simplifications for particular situations in which the magnitudes of the three parameters δ, θ, ε may be compared. It should be noted that θ depends only on the geometry of the plate. The typical length L and hence δ will depend on the smaller of a distance to the edge and the typical wavelength of the loading functions. There is little loss in assuming that $\delta \geq \theta$. The parameter ε will depend on the external loading. We see that the normal stress resultants will be of order $Eh+h\varepsilon\gamma^2$, the shear resultants of order $Wh\delta+h\varepsilon\gamma^2\delta$ and the bending moments of order $h^2W+h^2\varepsilon\gamma^2$. We also recall that Q^3 and P^3 are of order $\varepsilon\gamma^2$, P^α, Q^α of order $\varepsilon\delta$. Having determined δ it should be possible to choose ε and also make a comparison between E and W.

We now consider the consequences of various inequalities between the three small parameters $\varepsilon, \theta, \delta$.

We note first that if $\varepsilon = \theta = \delta^2$ then no further simplification of the equations (4.8), (4.10) is possible. The cases $E = \varepsilon\gamma^2$ or $W = \varepsilon\gamma^2$ are not considered here because of the need for higher order terms in the constitutive equations.

Linear theories

If $\varepsilon \ll \gamma^2$ we obtain linear theories. We take $\varepsilon = \gamma^4$ and we obtain the equations

$$\begin{aligned}
\delta N^{\alpha\beta}|_\beta + \tfrac{1}{2}P^\alpha &= O(\varepsilon\gamma^2\delta)\\
\tfrac{1}{3}\delta^2 M^{\alpha\beta}|_{\alpha\beta} + \theta K_{\alpha\beta}N^{\alpha\beta} + \tfrac{1}{2}P^3 + \tfrac{1}{2}\delta Q^\alpha|_\alpha &= O(\varepsilon\gamma^4)\\
\mathrm{e}^{\alpha\beta}\,\mathrm{e}^{\sigma\tau}[\delta^2 E_{\alpha\tau|\sigma\beta} - \theta K_{\alpha\tau}W_{\sigma\beta}] &= O(\varepsilon\gamma^4)\\
\delta W^\alpha_{\gamma|\beta} - \delta W^\alpha_{\beta|\gamma} &= O(\varepsilon\gamma^2\delta).
\end{aligned}$$

These equations are the relevant equations if $\theta = \delta^2$. Further simplifications are possible when $\theta \ll \delta^2$ or $\theta \gg \delta^2$. If $\theta = \gamma^4$, then the equations become

$$\begin{aligned}
\delta N^{\alpha\beta}|_\beta + \tfrac{1}{2}P^\alpha &= O(\varepsilon\gamma^2\delta)\\
\tfrac{1}{3}\delta^2 M^{\alpha\beta}|_{\alpha\beta} + \tfrac{1}{2}P^3 + \tfrac{1}{2}\delta Q^\alpha|_\alpha &= O(\varepsilon\gamma^4)\\
\mathrm{e}^{\alpha\beta}\,\mathrm{e}^{\sigma\tau}\delta^2 E_{\alpha\tau|\sigma\beta} &= O(\varepsilon\gamma^4)\\
\delta W^\alpha_{\gamma|\beta} - \delta W^\alpha_{\beta|\gamma} &= O(\varepsilon\gamma^2\delta)
\end{aligned}$$

which are the linear plate equations.

If $\theta = \delta \gg \delta^2$ then we have $\gamma^2 = \theta = \delta$ and

$$\begin{aligned}
\gamma^2 N^{\alpha\beta}|_\beta + \tfrac{1}{2}P^\alpha &= O(\varepsilon\gamma^4)\\
\gamma^2 K_{\alpha\beta}N^{\alpha\beta} + \tfrac{1}{2}P^3 &= O(\varepsilon\gamma^4)\\
\gamma^2\,\mathrm{e}^{\alpha\beta}\,\mathrm{e}^{\sigma\tau}K_{\alpha\tau}W_{\sigma\beta} &= O(\varepsilon\gamma^4)\\
\gamma^2 W^\alpha_{\beta|\gamma} - \gamma^2 W^\alpha_{\gamma|\beta} &= O(\varepsilon\gamma^4).
\end{aligned}$$

The system is uncoupled, the first two equations being the equations of a linear membrane theory.

In the cases when $W = \varepsilon\gamma^2$ or $E = \varepsilon\gamma^2$ we must use the refined equations (5.1), (5.3) and (5.4), (5.6). When $W = \varepsilon\gamma^2, \theta \geq \delta^2$ we need the full refined equations (5.1), (5.3) if $\theta = \gamma^4 \ll \delta^2$ we obtain the equations of generalized plane stress

$$\begin{aligned}
\gamma^2\,\mathrm{e}^{\alpha\beta}\,\mathrm{e}^{\sigma\tau}E_{\alpha\tau|\sigma\beta} &= O(\varepsilon\gamma^6)\\
\delta N^{\alpha\beta}|_\beta + \tfrac{1}{2}P^\alpha &= O(\varepsilon\gamma^2\delta).
\end{aligned}$$

When $E = \varepsilon\gamma^2, \theta \geq \delta^2$ we need the full refined equations (5.4), (5.6). If $\theta = \gamma^4 \ll \delta^2$ we obtain the plate bending equations

$$\begin{aligned}
\tfrac{1}{3}\delta^2 M^{\sigma\tau}|_{\sigma\tau} + \tfrac{1}{2}P^3 + \tfrac{1}{2}\delta Q^\alpha|_\alpha &= O(\varepsilon\gamma^6)\\
\delta W^\alpha_{\beta|\gamma} - \delta W^\alpha_{\gamma|\beta} &= O(\varepsilon\gamma^2\delta).
\end{aligned}$$

The linear theories are dealt with in the paper of Green [2].

Nonlinear theories

When $\varepsilon = \gamma^2$ we obtain nonlinear theories. As we have already noted when $\varepsilon = \theta = \delta^2$ we need the full equations (4.8), (4.10).

When $\varepsilon = \delta^2 = \gamma^2$ and $\theta \leq \gamma^4$ we obtain

$$\begin{aligned}
\delta N^{\alpha\beta}|_\beta + \tfrac{1}{2}P^\alpha &= O(\varepsilon\gamma^2\delta)\\
\tfrac{1}{3}\delta^2 M^{\alpha\beta}|_{\alpha\beta} + \tfrac{1}{2}P^3 + \tfrac{1}{2}\delta Q^\alpha|_\alpha - W_{\alpha\beta}N^{\alpha\beta} &= O(\varepsilon\gamma^4)\\
e^{\alpha\beta}\, e^{\sigma\tau}[\delta^2 E_{\alpha\tau|\sigma\beta} + \tfrac{1}{2}W_{\alpha\tau}W_{\sigma\beta}] &= O(\varepsilon\gamma^4)\\
\delta W^\alpha_{\beta|\gamma} - \delta W^\alpha_{\gamma|\beta} &= O(\varepsilon\gamma^2\delta)
\end{aligned}$$

which are the von Karman plate equations.

When $\varepsilon = \theta = \delta \gg \delta^2$ we have

$$\begin{aligned}
\gamma^2 N^{\alpha\beta}|_\beta + \tfrac{1}{2}P^\alpha &= O(\varepsilon\gamma^4)\\
\gamma^2 K_{\beta\gamma}N^{\beta\gamma} - W_{\beta\gamma}N^{\beta\gamma} + \tfrac{1}{2}P^3 &= O(\varepsilon\gamma^4)\\
e^{\alpha\beta}\, e^{\sigma\tau}[-\gamma^2 K_{\alpha\tau}W_{\sigma\beta} + \tfrac{1}{2}W_{\alpha\tau}W_{\sigma\beta}] &= O(\varepsilon\gamma^4)\\
\gamma^2[W^\alpha_{\beta|\gamma} - W^\alpha_{\gamma|\beta}] &= O(\varepsilon\gamma^4).
\end{aligned}$$

The last two equations determine W and the first two are of a nonlinear membrane type.

We examine finally the case $\varepsilon \gg \theta$, $\varepsilon \gg \delta^2$. The equations are not altered until $\theta \leq \gamma^4$, $\delta \leq \gamma^2$ when we get

$$\begin{aligned}
\delta N^{\alpha\beta}|_\beta + \tfrac{1}{2}P^\alpha &= O(\varepsilon\gamma^2\delta)\\
W_{\alpha\beta}N^{\alpha\beta} - \tfrac{1}{2}P^3 &= O(\varepsilon\gamma^4)\\
W^\tau_\sigma W^\sigma_\tau - W^\tau_\tau W^\sigma_\sigma &= O(\varepsilon\gamma^4)\\
\delta(W^\alpha_{\beta|\gamma} - W^\alpha_{\gamma|\beta}) &= O(\varepsilon\gamma^2\delta).
\end{aligned}$$

It should be recalled that δ depends on the distance to the boundary and therefore in any theory for which we have taken $\delta \ll \gamma$ the equations will not be valid for distances less than h/γ from the edge. To obtain solutions for the whole region one must take $\delta = \gamma$. Thus only the equations for which $\gamma = \delta$ or $\theta = \delta^2$, $\varepsilon = \delta^2$ or $\theta \ll \delta^2$ and $\varepsilon \ll \delta^2$ are relevant for boundary value problems. The equations must of course be combined with suitable boundary conditions which may be complicated by nonlinear effects. To obtain the correct boundary conditions for this interior problem we would have to consider a boundary layer theory. This we postpone for further study but we believe that the boundary conditions would be similar to those of the linear theory.

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Абстракт—Путем обыкновенного асимптотического подхода определяется теория малых деформаций для оболочек, исходя из трехмерных уравнений равновесия и совместности. Уравнения наиболее высо кого порядка согласовывуются с такими же уравнениями Койтера [9] и Джона [12].

При выводе этих уравнений появлтются три малых параметра и получаются разные специальные теории, при разных предположениях относительно величин параметров.

Для случаев, когда как напряжения расширения, так и напряжения изгибф оказываются преобладающими, надо применить специальное уточнение теории. Они получаются дла линейного случая и согласовывуются с уточненными уравнениями Джона для внутренной области [13, 14].